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Parametric Duality Models for Discrete Minmax Fractional Programming Problems on Higher Order Invex Functions

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Abstract. This communication deals with a new second-order parametric duality theory for a discrete minmax fractional programming problem relating to second-order necessary and sufficient optimality conditions. First, some second-order duality models are formulated, and then some weak, strong, and strictly converse duality theorems are established applying new classes of generalized second-order invex functions.

Key Words: Discrete minmax fractional programming, Generalized second-order $(\phi, \eta, \zeta, \rho, \theta, m)$-invex functions, Second-order duality models, Duality theorems.
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1 Introduction

In this communication, we consider the following discrete minmax fractional programming problem:

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$$(P) \qquad \text{Minimize} \max_{1 \le i \le p} \frac{f_i(x)}{g_i(x)}$$

subject to $\quad G_j(x) \le 0,\ j \in \underline{q}, \quad H_k(x) = 0,\ k \in \underline{r}, \quad x \in X,$

where X is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), f_i, g_i, $i \in \underline{p} \equiv \{1, 2, \ldots, p\}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are real-valued functions defined on X, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all x satisfying the constraints of (P).

In this paper, we intend to explore and examine some new classes of generalized second-order invex functions (referred to as sonvex functions) in order to establish a set of second-order necessary optimality conditions, and numerous sets of second-order sufficient optimality conditions based on various generalized $(\phi, \eta, \zeta, \rho, \theta, m)$-sonvexity assumptions. Furthermore, we shall construct two second-order parametric duality models and prove some weak, strong, and strict converse duality theorems applying a variety of $(\phi, \eta, \zeta, \rho, \theta, m)$-sonvexity hypotheses. The paper is organized as follows: In Section 2, we introduce a class of new basic definitions and recall some auxiliary results that will be used in the sequel. In Section 3, we consider two second-order parametric duality models for (P) with relatively newer constraint structures and prove weak, strong, and strict converse duality theorems using various generalized second-order $(\phi, \eta, \zeta, \rho, \theta, m)$-invexity assumptions. In Section 4, we summarize our main results and also point out (based on our observations) some research opportunities arising from certain modifications of the principal minmax model investigated in the present communication. For more related details, we refer the reader [1 - 40].

We remark that all the duality results obtained for (P) are also applicable, when appropriately specialized, to the following three classes of problems with discrete minmax, fractional, and conventional objective functions, which are particular cases of (P):

$$(P1) \qquad \underset{x \in \mathbb{F}}{\text{Minimize}} \max_{1 \le i \le p} f_i(x);$$

$$(P2) \qquad \underset{x \in \mathbb{F}}{\text{Minimize}} \frac{f_1(x)}{g_1(x)};$$

$$(P3) \qquad \operatorname*{Minimize}_{x\in\mathbb{F}} \; f_1(x),$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \{x \in X : G_j(x) \le 0,\ j \in \underline{q},\quad H_k(x) = 0,\quad k \in \underline{r}\}.$$

2 Preliminaries

In this section we introduce certain classes of generalized convex functions, which encompass most of generalized notions for convex functions in the literature. Let $f : X \to \mathbb{R}$ be a twice differentiable function.

Definition 2.1 *The function f is said to be* (strictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-sonvex *at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R},\ \eta, \zeta : X \times X \to \mathbb{R}^n,\ \rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,*

$$\begin{aligned} &\phi\big(f(x) - f(x^*)\big) \\ (>) \ge \quad &\langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle \zeta(x, x^*), \nabla^2 f(x^*) z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^m, \end{aligned}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

The function f is said to be (strictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-sonvex *on X if it is* (strictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-sonvex *at each $x^* \in X$.*

Definition 2.2 *The function f is said to be* (strictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-pseudosonvex *at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R},\ \eta, \zeta : X \times X \to \mathbb{R}^n,\ \rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,*

$$\begin{aligned} &\langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle \zeta(x, x^*), \nabla^2 f(x^*) z\rangle \ge -\rho(x, x^*)\|\theta(x, x^*)\|^m \\ \Rightarrow\quad &\phi\big(f(x) - f(x^*)\big)\ (>) \ge 0. \end{aligned}$$

The function f is said to be (strictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-pseudosonvex *on X if it is* (strictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-pseudosonvex *at each $x^* \in X$.*

Definition 2.3 *The function f is said to be* (prestrictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-quasisonvex *at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta, \zeta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ and $z \in \mathbb{R}^n$,*

$$\begin{aligned} & \phi\big(f(x) - f(x^*)\big)\,(<) \leq 0 \\ \Rightarrow \quad & \langle \nabla f(x^*), \eta(x, x^*) \rangle + \frac{1}{2} \langle \zeta(x, x^*), \nabla^2 f(x^*) z \rangle \leq -\rho(x, x^*) \|\theta(x, x^*)\|^m. \end{aligned}$$

The function f is said to be (prestrictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-quasisonvex *on X if it is* (prestrictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-quasisonvex *at each $x^* \in X$.*

From the above definitions it is clear that if f is $(\phi, \eta, \zeta, \rho, \theta, m)$-sonvex at x^*, then it is both $(\phi, \eta, \zeta, \rho, \theta, m)$-pseudosonvex and $(\phi, \eta, \zeta, \rho, \theta, m)$-quasisonvex at x^*, if f is $(\phi, \eta, \zeta, \rho, \theta, m)$-quasisonvex at x^*, then it is prestrictly $(\phi, \eta, \zeta, \rho, \theta, m)$-quasisonvex at x^*, and if f is strictly $(\phi, \eta, \zeta, \rho, \theta, m)$-pseudosonvex at x^*, then it is $(\phi, \eta, \zeta, \rho, \theta, m)$-quasisonvex at x^*.

In the proofs of the duality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\phi, \eta, \zeta, \rho, \theta, m)$-quasisonvexity can be defined in the following equivalent way:

The function f is said to be $(\phi, \eta, \zeta, \rho, \theta, m)$-*quasisonvex* at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta, \zeta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} & \langle \nabla f(x^*), \eta(x, x^*) \rangle + \frac{1}{2} \langle \zeta(x, x^*), \nabla^2 f(x^*) z \rangle > -\rho(x, x^*) \|\theta(x, x^*)\|^m \\ \Rightarrow \quad & \phi\big(f(x) - f(x^*)\big) > 0. \end{aligned}$$

Thus, the new classes of generalized convex functions specified in Definitions 2.1 - 2.3 contain a variety of special cases that can easily be identified by appropriate choices of ϕ, η, ζ, ρ, θ, and m.

Next, we consider an interesting example that may connect fractional derivatives [1] with higher order mathematical programming.

Example 2.1 *Let $f : X \to \mathbb{R}$ be a function. Then function f is said to be $(\phi, \eta, \rho, \theta, m)$-invex at x^* for the left Caputo fractional partial derivative of order α, $\alpha \geq 1$ if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer m such that for each $x \in X$ $(x \neq x^*)$,*

$$\phi\Big([f(x) - f(x^*)]\Big) \geq \big\langle \nabla_\alpha^{++} f(x^*), \eta(x, x^*)\big\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^m,$$

where

$$\nabla_\alpha^{++} f(x^*) = \Big(\frac{\delta^\alpha f(x^*)}{\delta {x_1}^\alpha}, \cdots, \frac{\delta^\alpha f(x^*)}{\delta x_n^\alpha}\Big).$$

We next recall a set of second-order necessary optimality conditions for (P). The following result will be needed for proving strong and strict converse duality theorems.

Theorem 2.1 [34] *Let x^* be an optimal solution of (P), let $\lambda^* = \varphi(x^*) \equiv \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)\}$, and assume that the functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are twice continuously differentiable at x^*, and that the second-order Guignard constraint qualification holds at x^*. Then for each $z^* \in C(x^*)$, there exist*

$$u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0, \ \sum_{i=1}^{p} u_i = 1\},$$

$v^* \in \mathbb{R}_+^q \equiv \{v \in \mathbb{R}^q : v \geq 0\}$, and $w^* \in \mathbb{R}^r$ such that

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0,$$

$$\Big\langle z^*, \Big\{ \sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*) \Big\} z^* \Big\rangle \geq 0,$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \ \ i \in \underline{p}, \quad v_j^* G_j(x^*) = 0, \ \ j \in \underline{q},$$

where $C(x^*)$ is the set of all critical directions of (P) at x^*, that is,

$$C(x^*) = \Big\{ z \in \mathbb{R}^n \colon \langle \nabla f_i(x^*) - \lambda \nabla g_i(x^*), z \rangle = 0, \ \ i \in A(x^*), \langle \nabla G_j(x^*), z \rangle \leq 0, \ j \in B(x^*), \langle \nabla H_k(x^*), z \rangle = 0, \ \ k \in \underline{r} \Big\}$$

for

$$A(x^*) = \{ j \in \underline{p} : f_j(x^*)/g_j(x^*) = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*) \},$$

$$\text{and} \qquad B(x^*) = \{ j \in \underline{q} : G_j(x^*) = 0 \}.$$

For brevity, we shall henceforth refer to x^* as a *normal* optimal solution of (P) if it is an optimal solution and satisfies the second-order Guignard constraint qualification.

Now we shall assume that the functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are twice continuously differentiable on the open set X. Moreover, we shall assume, without loss of generality, that $g_i(x) > 0$, $i \in \underline{p}$, and $\varphi(x) \geq 0$ for all $x \in X$.

3 Duality Models

In this section, we consider a duality model with relative constraint structures and prove weak, strong, and strictly converse duality theorems.

Consider the following two duality problems:

(DI) Maximize λ

subject to

$$\sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^{q} v_j \nabla G_j(y) + \sum_{k=1}^{r} w_k \nabla H_k(y) = 0, \tag{3.1}$$

$$\Big\langle \zeta(x,y), \Big\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j=1}^{q} v_j \nabla^2 G_j(y) + \sum_{k=1}^{r} w_k \nabla^2 H_k(y) \Big\} z \Big\rangle \geq 0, \tag{3.2}$$

for all $x \in \mathbb{F}$,

$$\sum_{i=1}^{p} u_i[f_i(y) - \lambda g_i(y)] + \sum_{j=1}^{q} v_j G_j(y) + \sum_{k=1}^{r} w_k H_k(y) \geq 0, \tag{3.3}$$

$$y \in X,\ z \in C(y),\ u \in U,\ v \in \mathbb{R}^q_+,\ w \in \mathbb{R}^r,\ \lambda \in \mathbb{R}_+. \tag{3.4}$$

$(\tilde{D}I)$ Maximize λ
subject to (3.2) - (3.4) and

$$\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^{q} v_j \nabla G_j(y) + \sum_{k=1}^{r} w_k \nabla H_k(y), \eta(x,y) \Big\rangle \geq 0, \tag{3.5}$$

for all $x \in \mathbb{F}$.

Comparing (DI) and $(\tilde{D}I)$, we see that $(\tilde{D}I)$ is relatively more general than (DI) in the sense that any feasible solution of (DI) is also feasible for $(\tilde{D}I)$, but the converse is not necessarily true. Furthermore, we observe that (3.1) is a system of n equations, whereas (3.5) is a single inequality. Clearly, from a computational point of view, (DI) is preferable to $(\tilde{D}I)$ because of the dependence of (3.5) on the feasible set of (P).

Despite these apparent differences, it turns out that the statements and proofs of all the duality theorems for $(P)-(DI)$ and $(P)-(\tilde{D}I)$ are almost identical and, therefore, we shall consider only the pair $(P)-(DI)$.

For the sake of the completeness, we shall use the following list of symbols in the statements and proofs of our duality theorems:

$$\begin{aligned}
\mathcal{C}(x,v) &= \sum_{j=1}^{q} v_j G_j(x),\\
\mathcal{D}_k(x,w) &= w_k H_k(x), \quad k \in \underline{r},\\
\mathcal{D}(x,w) &= \sum_{k=1}^{r} w_k H_k(x),\\
\mathcal{E}_i(x,\lambda) &= f_i(x) - \lambda g_i(x), \quad i \in \underline{p},\\
\mathcal{E}(x,u,\lambda) &= \sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)],\\
\mathcal{G}(x,v,w) &= \sum_{j=1}^{q} v_j G_j(x) + \sum_{k=1}^{r} w_k H_k(x),
\end{aligned}$$

$I_+(u) = \{i \in \underline{p} \colon u_i > 0\}$, $J_+(v) = \{j \in \underline{q} \colon v_j > 0\}$, $K_*(w) = \{k \in \underline{r} \colon w_k \neq 0\}$.

In the proofs of our duality theorems, we shall make frequent use of the following auxiliary result which provides an alternative expression for the objective function of (P).

Lemma 3.1 [36] *For each $x \in X$,*

$$\varphi(x) = \max_{1 \le i \le p} \frac{f_i(x)}{g_i(x)} = \max_{u \in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)}.$$

In next two theorems, we show that (DI) is a dual problem for (P), and then other theorems deal with th strong duality, and strictly converse duality theorems.

Theorem 3.1 *(Weak Duality) Let x and $\mathcal{S} \equiv (y,z,u,v,w,\lambda)$ be arbitrary feasible solutions of (P) and (DI), respectively, and assume that either one* *of the following two sets of hypotheses is satisfied:*

(a) (i) *for each* $i \in I_+ \equiv I_+(u)$, f_i *is* $(\phi, \eta, \zeta, \bar{\rho}_i, \theta, m)$*-sonvex at* y*;*

(ii) *for each* $i \in I_+ \equiv I_+(u)$, $-g_i$ *is* $(\phi, \eta, \zeta, \tilde{\rho}_i, \theta, m)$*-sonvex at* y*;*

(iii) *for each* $j \in J_+ \equiv J_+(v)$, G_j *is* $(\phi, \eta, \zeta, \hat{\rho}_j, \theta, m)$*-sonvex at* y*;*

(iv) *for each* $k \in K_* \equiv K_*(w)$, $w_k H_k$ *is* $(\phi, \eta, \zeta, \breve{\rho}_k, \theta, m)$*-sonvex at* y*;*

(v) ϕ *is superlinear and* $\phi(a) \geq 0 \Rightarrow a \geq 0$*;*

(vi) $\sum_{i \in I_+} u_i[\bar{\rho}_i(x,y) + \lambda \tilde{\rho}_i(x,y)] + \sum_{j \in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k \in K_*} \breve{\rho}_k(x,y) \geq 0$*;*

(b) *the Lagrangian-type function*

$$\xi \to L(\xi, u, v, w, \lambda) = \sum_{i=1}^{p} u_i[f_i(\xi) - \lambda g_i(\xi)] + \sum_{j=1}^{q} v_j G_j(\xi) + \sum_{k=1}^{r} w_k H_k(\xi)$$

is $(\phi, \eta, \zeta, \rho, \theta, m)$*-pseudosonvex at* y*,* $\rho(x,y) \geq 0$*, and* $\phi(a) \geq 0 \Rightarrow a \geq 0$*.*

Then $\varphi(x) \geq \lambda$.

Proof. (a): Using the hypotheses specified in (i) - (iv), we have

$$\phi\big(f_i(x) - f_i(y)\big) \geq \langle \nabla f_i(y), \eta(x,y) \rangle + \frac{1}{2}\langle \zeta(x,y), \nabla^2 f_i(y) z \rangle + \bar{\rho}_i(x,y) \|\theta(x,y)\|^m, \quad i \in I_+, \tag{3.6}$$

$$\phi\big(-g_i(x) + g_i(y)\big) \geq \langle -\nabla g_i(y), \eta(x,y) \rangle - \frac{1}{2}\langle \zeta(x,y), \nabla^2 g_i(y) z \rangle + \tilde{\rho}_i(x,y) \|\theta(x,y)\|^m, \quad i \in I_+, \tag{3.7}$$

$$\phi\big(G_j(x) - G_j(y)\big) \geq \langle \nabla G_j(y), \eta(x,y) \rangle + \frac{1}{2}\langle \zeta(x,y), \nabla^2 G_j(y) z \rangle + \hat{\rho}_j(x,y) \|\theta(x,y)\|^m, \quad j \in J_+, \tag{3.8}$$

$$\begin{aligned}(3.9) \qquad & \phi\big(w_k H_k(x) - w_k H_k(y)\big) \\ \geq\ & \langle w_k \nabla H_k(y), \eta(x,y)\rangle + \frac{1}{2}\langle \zeta(x,y), w_k \nabla^2 H_k(y) z\rangle \\ & + \breve{\rho}_k(x,y)\|\theta(x,y)\|^m, \quad k \in K_*.\end{aligned}$$

Now, multiplying (3.6) by u_i and then summing over $i \in \underline{p}$, (3.7) by λu_i and then summing $i \in \underline{p}$, (3.8) by v_j and then summing over $j \in \underline{q}$, summing (3.9) over $k \in \underline{r}$, adding the resulting inequalities, using the superlinearity of ϕ, and setting $u_i = 0,\ i \notin I_+,\ v_j = 0,\ j \notin J_+$, and $w_k = 0,\ k \notin K_*$, we obtain

$$\begin{aligned}&\phi\Big(\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] + \sum_{j=1}^{q} v_j G_j(x) + \sum_{k=1}^{r} w_k H_k(x) \\ &\qquad - \Big\{\sum_{i=1}^{p} u_i[f_i(y) - \lambda g_i(y)] + \sum_{j=1}^{q} v_j G_j(y) + \sum_{k=1}^{r} w_k H_k(y)\Big\}\Big) \\ &\geq \Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^{q} v_j \nabla G_j(y) + \sum_{k=1}^{r} w_k \nabla H_k(y), \eta(x,y)\Big\rangle + \\ &\frac{1}{2}\Big\langle \zeta(x,y), \Big\{\sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j=1}^{q} v_j \nabla^2 G_j(y) + \sum_{k=1}^{r} w_k \nabla^2 H_k(y)\Big\} z\Big\rangle \\ &+ \Big\{\sum_{i\in I_+} u_i[\bar{\rho}_i(x,y) + \lambda \tilde{\rho}_i(x,y)] + \sum_{j \in J_+} v_j \hat{\rho}_j(x,y) + \sum_{k\in K_*} \breve{\rho}_k(x,y)\Big\}\|\theta(x,y)\|^m.\end{aligned}$$

Because of the dual feasibility of $\mathcal{S}$, (3.1), (3.2), and (vi), the above inequality becomes

$$\begin{aligned}&\phi\Big(\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] + \sum_{j=1}^{q} v_j G_j(x) \\ &+ \sum_{k=1}^{r} w_k H_k(x) - \Big\{\sum_{i=1}^{p} u_i[f_i(y) - \lambda g_i(y)] + \sum_{j=1}^{q} v_j G_j(y) + \sum_{k=1}^{r} w_k H_k(y)\Big\}\Big) \geq 0.\end{aligned}$$

But $\phi(a) \geq 0 \Rightarrow a \geq 0$, and hence

$$\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] + \sum_{j=1}^{q} v_j G_j(x)$$
$$+\sum_{k=1}^{r} w_k H_k(x) - \Big\{ \sum_{i=1}^{p} u_i[f_i(y) - \lambda g_i(y)] + \sum_{j=1}^{q} v_j G_j(y) + \sum_{k=1}^{r} w_k H_k(y) \Big\} \geq 0.$$

In view of the primal feasibility of x and (3.3), this inequality further reduces to

$$\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] \geq 0. \tag{3.10}$$

Now using (3.10) and Lemma 3.1, we obtain the weak duality inequality as follows:

$$\varphi(x) = \max_{a \in U} \frac{\sum_{i=1}^{p} a_i f_i(x)}{\sum_{i=1}^{p} a_i g_i(x)} \geq \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)} \geq \lambda.$$

(b): Using the dual feasibility of $\mathcal{S}$, nonnegativity of $\rho(x, y)$, (3.1), and (3.2), we obtain the following inequality:

$$\begin{aligned}&\langle \nabla L(y, u, v, w, \lambda), \eta(x, y)\rangle + \frac{1}{2}\big\langle \zeta(x, y), \nabla^2 L(y, u, v, w, \lambda) z\big\rangle \ \geq \ 0 \\ &\geq \ -\rho(x, y)\|\theta(x, y)\|^m,\end{aligned}$$

which in view of our $(\phi, \eta, \zeta, \rho, \theta, m)$-pseudosonvexity assumption implies that

$$\phi\big(L(x, u, v, w, \lambda) - L(y, u, v, w, \lambda)\big) \geq 0.$$

Since $\phi(a) \geq 0 \Rightarrow a \geq 0$, we have

$$L(x, u, v, w, \lambda) - L(y, u, v, w, \lambda) \geq 0.$$

Because $x \in \mathbb{F},\ v \geq 0$, and (3.3) holds, we get

$$\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] \geq 0,$$

which is precisely (3.10). As seen in the proof of part (a), this inequality leads to the desired conclusion that $\varphi(x) \geq \lambda$. ∎

Theorem 3.2 *(Strong Duality) Let x^* be a normal optimal solution of (P), let $\lambda^* = \varphi(x^*)$, and assume that either one of the two sets of conditions specified in Theorem 3.1 is satisfied for all feasible solutions of (DI). Then for each $z^* \in C(x^*)$, there exist $u^* \in U$, $v^* \in \mathbb{R}^q_+$, and $w^* \in \mathbb{R}^r$ such that $\mathcal{S}^* \equiv (x^*, z^*, u^*, v^*, w^*, \lambda^*)$ is an optimal solution of (DI) and $\varphi(x^*) = \lambda^*$.*

Proof. Since x^* is a normal optimal solution of (P), by Theorem 2.1, for each $z^* \in C(x^*)$, there exist u^*, v^*, w^*, and $\lambda^*(= \varphi(x^*))$, as specified above, such that $\mathcal{S}^*$ is a feasible solution of (DI). If $\mathcal{S}^*$ were not optimal, then there would exist a feasible solution $(\tilde{y}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\lambda})$ of (DI) such that $\tilde{\lambda} > \lambda^* = \varphi(x^*)$ contradicting Theorem 3.1. Therefore, $\mathcal{S}^*$ is an optimal solution of (DI). ∎

We also have the following converse duality result for (P) and (DI).

Theorem 3.3 *(Strict Converse Duality) Let x^* be a normal optimal solution of (P), let $\tilde{\mathcal{S}} \equiv (\tilde{x}, \tilde{z}, \tilde{\lambda}, \tilde{u}, \tilde{v}, \tilde{w})$ be an optimal solution of (DI), and assume that either one of the following two sets of hypotheses is satisfied:*

(a) The assumptions of part (a) of Theorem 3.1 are satisfied for the feasible solution $\tilde{\mathcal{S}}$ of (DI), $\phi(a) > 0 \Rightarrow a > 0$, and f_i is strictly $(\phi, \eta, \zeta, \bar{\rho}_i, \theta, m)$-sonvex at $\tilde{x}$ for at least one index $i \in I_+(\tilde{u})$, or $-g_i$ is strictly $(\phi, \eta, \zeta, \tilde{\rho}_i, \theta, m)$-sonvex at $\tilde{x}$ for at least one index $i \in I_+(\tilde{u})$ (and $\tilde{\lambda} > 0$), or G_j is strictly $(\phi, \eta, \zeta, \hat{\rho}_j, \theta, m)$-sonvex at $\tilde{x}$ for at least one index $j \in J_+(\tilde{v})$, or $\mathcal{D}_k(\cdot, \tilde{w})$ is strictly $(\phi, \eta, \zeta, \breve{\rho}_k, \theta, m)$-sonvex at $\tilde{x}$ for at least one index $k \in K_*(\tilde{w})$, or $\sum_{i \in I_+(\tilde{u})} \tilde{u}_i[\bar{\rho}_i(x^*, \tilde{x}) + \tilde{\lambda}\tilde{\rho}_i(x^*, \tilde{x})] + \sum_{j \in J_+(\tilde{v})} \tilde{v}_j \hat{\rho}_j(x^*, \tilde{x}) + \sum_{k \in K_*(\tilde{w})} \breve{\rho}_k(x^*, \tilde{x}) > 0$.

(b) The Lagrangian-type function $\omega \rightarrow L(\omega, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\lambda})$ is strictly $(\phi, \eta, \zeta, \rho, \theta, m)$-pseudosonvex at $\tilde{x}$, $\rho(x^*, \tilde{x}) \geq 0$, and $\phi(a) > 0 \Rightarrow a > 0$.

Then $\tilde{x} = x^*$, that is, $\tilde{x}$ is an optimal solution of (P), and $\varphi(x^*) = \tilde{\lambda}$.

Proof. (a) : Since x^* is a normal optimal solution of (P), by Theorem 2.1, there exist $z^* \in \mathbb{R}^n,\ u^* \in U,\ v^* \in \mathbb{R}^q_+,\ w^* \in \mathbb{R}^r$, and $\lambda^*(=\varphi(x^*))$ such that $\mathcal{S}^* \equiv (x^*, z^*, u^*, v^*, w^*, \lambda^*)$ is an optimal solution of (DI) and $\varphi(x^*) = \lambda^*$. Suppose to the contrary that $\tilde{x} \neq x^*$. Now proceeding as in the proof of Theorem 3.1 (with x replaced by x^* and $\mathcal{S}$ by $\tilde{\mathcal{S}}$) and using any of the conditions set forth above, we arrive at the strict inequality

$$\sum_{i=1}^{p} \tilde{u}_i[f_i(x^*) - \tilde{\lambda} g_i(x^*)] > 0.$$

Now using this inequality in conjunction with Lemma 3.1, as in the proof of part (a) of Theorem 3.1, we arrive at the strict inequality $\varphi(x^*) > \tilde{\lambda}$ which contradicts the fact that $\varphi(x^*) = \lambda^* = \tilde{\lambda}$. Therefore, we conclude that $\tilde{x} = x^*$ and $\varphi(x^*) = \tilde{\lambda}$.

(b) : The proof is similar to that of part (a). ■

4 Concluding Remarks

Based on a Dinkelbach-type [6] parametric approach, we have formulated second-order parametric duality model for a discrete minmax fractional programming problem and established a multiplicity of duality theorems using a variety of generalized $(\phi, \eta, \zeta, \rho, \theta, m)$-sonvexity assumptions. Furthermore, the results obtained here seem to be useful for investigating some other aspects of nonlinear programming problems, especially the sufficient optimality and duality aspects of the following semiinfinite minmax fractional programming problem:

$$\text{Minimize } \max_{1 \le i \le p} \frac{f_i(x)}{g_i(x)}$$

subject to

$$G_j(x,t) \le 0 \ \text{ for all } t \in T_j, \ \ j \in \underline{q},$$

$$H_k(x,s) = 0 \ \text{ for all } s \in S_k, \ \ k \in \underline{r},$$

$$x \in X,$$

where X, f_i, and g_i, $i \in \underline{p}$, are as defined in the description of (P), for each $j \in \underline{q}$ and $k \in \underline{r}$, T_j and S_k are compact subsets of complete metric spaces, for each $j \in \underline{q}$, $\xi \to G_j(\xi, t)$ is a real-valued function defined on X for all $t \in T_j$, for each $k \in \underline{r}$, $\xi \to H_k(\xi, s)$ is a real-valued function defined on X for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $t \to G_j(x, t)$ and $s \to H_k(x, s)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$.

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