On \((h, k)\)-trichotomy for evolution operators in Banach spaces

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Abstract. The paper considers three general trichotomy concepts, which are generalizations of the classical exponential and polynomial trichotomies. Characterizations and connections between these concepts are given.

Key Words: Evolution operators, \((h,k)\)trichotomy, strong \((h,k)\)-trichotomy, weak \((h,k)\)-trichotomy

MSC 2000: 34D05, 34D09

1 Introduction

As natural generalizations of the dichotomy properties, the concepts of trichotomy introduced by R. J. Sacker and G. R. Sell in [15] play a crucial role in the qualitative theory of dynamical systems. The first studies devoted to the trichotomic behaviors for differential equations were initiated by S.
Elaydi and O. Hayek in [4] and [5].

In the last decades, a substantial part of the asymptotic theory of differential equations was devoted to the extension of the methods used in dichotomy theory to the trichotomy case (see [7], [8], [9], [12], [13], [14]). Characterizations for the uniform exponential trichotomy of evolution operators in Banach spaces was obtained by M. Megan and C. Stoica in [10], [11] and M. I. Kovács in [6].

In the nonautonomous setting the concept of uniform exponential (or polynomial) trichotomy is too restrictive and it is important to look for more general behaviors, for example the nonuniform case, where a consistent contribution is due to L. Barreira and C. Valls (see [1], [2], [3]).

A new perspective on the discrete input-output techniques in the detection of the trichotomic behavior of nonautonomous dynamical systems is presented by B. Sasu and A.L. Sasu in [16].

This paper considers three general trichotomy concepts with growth rates given by increasing functions. These new trichotomies include the traditional (uniform or nonuniform) exponential or polynomial trichotomy. Thus we obtain a systematic classification of trichotomy concepts with the connections between them.

2 Definitions, notations and preliminary results

Let $X$ be a real or complex Banach space and let $B(X)$ be the algebra Banach of all bounded linear operators on $X$. The norm on $X$ and on $B(X)$ will be denoted by $\| \cdot \|$. Denote by $I$ the identity operator on $X$.

Let $\Delta$ be the set of all pairs $(t, s)$ of real numbers with $t \geq s \geq 0$. We denote by $T = \Delta \times X$.

In this section we give some preliminary definitions.

**Definition 2.1** An operator valued function $U : \Delta \to B(X)$ is said to be an evolution operator on $X$ if

$(e_1)$ $U(t, t) = I$ for every $t \geq 0$;

$(e_2)$ $U(t, s)U(s, t_0) = U(t, t_0)$ for all $(t, s) \in \Delta$ and $(s, t_0) \in \Delta$.

If the condition $(e_2)$ holds for all $t, s, t_0 \geq 0$ then we say that $U$ is a reversible evolution operator.
Definition 2.2  An operator valued function $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is called a family of projections on $X$ if

$$P(t)^2 = P(t), \quad \text{for every } t \geq 0.$$ 

Definition 2.3  Given an evolution operator $U : \Delta \to \mathcal{B}(X)$, we say that a family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is invariant for $U$ if

$$P(t)U(t, s) = U(t, s)P(s)$$

for all $(t, s) \in \Delta$.

Definition 2.4  Three families of projections $P_1, P_2, P_3 : \mathbb{R}_+ \to \mathcal{B}(X)$ are called supplementary if for every $t \in \mathbb{R}_+$ we have

1. $P_1(t) + P_2(t) + P_3(t) = I$;
2. $P_i(t)P_j(t) = 0$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$.

If $\mathcal{P} = \{P_1, P_2, P_3\}$ is a family of three supplementary projections which are invariant for $U$, then we say that the pair $(U, \mathcal{P})$ is a trichotomic pair.

Definition 2.5  Let $P : \mathbb{R}_+ \to \mathcal{B}(X)$ be a family of projections on $X$ which is invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$. We say that $P$ is strongly invariant for $U$ if for all $(t, s) \in \Delta$ the restriction of $U(t, s)$ on $\text{Range}P(s)$ is an isomorphism from $\text{Range}P(s)$ to $\text{Range}P(t)$.

Remark 2.1  If a family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is strongly invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ then there exists $V : \Delta \to \mathcal{B}(X)$ such that $V(t, s)$ is an isomorphism from $\text{Range}P(t)$ to $\text{Range}P(s)$ and

1. $U(t, s)V(t, s)P(t) = P(t)$;
2. $V(t, s)U(t, s)P(s) = P(s)$,

for all $(t, s) \in \Delta$.

The map $V$ is called the skew-evolution operator associated to the pair $(U, \mathcal{P})$. 
Definition 2.6 A family $\mathcal{P} = \{P_1, P_2, P_3\}$ of three supplementary projections is called compatible with the evolution operator $U : \Delta \to B(X)$ if

1. $P_1$ is invariant for $U$;
2. $P_2$ and $P_3$ are strongly invariant for $U$.

For an evolution operator $U : \Delta \to B(X)$ and $\mathcal{P}$ compatible with $U$, we will denote by $V_2(t, s)$ and $V_3(t, s)$ the skew-evolution operators associated to the pairs $(U, P_2)$ and $(U, P_3)$.

3 (h,k)-trichotomy

Let $h, k : \mathbb{R}^+ \to [1, \infty)$ be two increasing functions and $(U, \mathcal{P})$ a trichotomic pair.

Definition 3.1 We say that the pair $(U, \mathcal{P})$ is (h,k)-trichotomic and denote $(h, k) - t$, if there are the constants $N \geq 1, a > 0, b \geq 0, c > 0$ such that:

1. $h(t)^a \|U(t, s)P_1(s)x\| \leq Nh(s)^a k(s)^b \|P_1(s)x\|$;
2. $h(t)^a \|P_2(s)x\| \leq Nh(s)^a k(t)^b \|U(t, s)P_2(s)x\|$;
3. $h(s)^c \|U(t, s)P_3(s)x\| \leq Nh(t)^c k(s)^b \|P_3(s)x\|$;
4. $h(s)^c \|P_3(s)x\| \leq Nh(t)^c k(t)^b \|U(t, s)P_3(s)x\|$,

for all $(t, s, x) \in \Delta \times X$.

Remark 3.1 As particular cases of (h,k)-trichotomy we remark that

1. if $h(t) = k(t) = e^t$, then we recover the notion of nonuniform exponential trichotomy and in particular when the function $k$ is constant (or $b = 0$) we obtain the classical notion of uniform exponential trichotomy.
2. if $h(t) = k(t) = t + 1$, then we recover the notion of nonuniform polynomial trichotomy and in particular when the function $k$ is constant (or $b = 0$) we obtain the classical notion of uniform polynomial trichotomy.
(iii) if \( P_3 = 0 \) in Definition 3.1, then we recover the notion of \((h,k)\)-dichotomy.

(iv) if \( h(t) = k(t) = e^t \) and \( P_3 = 0 \) then we recover the notion of nonuniform exponential dichotomy and in particular when the function \( k \) is constant (or \( b = 0 \)) we obtain the classical notion of uniform exponential dichotomy.

(v) if \( h(t) = k(t) = t + 1 \) and \( P_3 = 0 \), then we recover the notion of nonuniform polynomial dichotomy and in particular when the function \( k \) is constant (or \( b = 0 \)) we obtain the classical notion of uniform polynomial dichotomy.

An example of a trichotomic pair \((U, P)\) which is \((h,k)\)-trichotomic is given below.

**Example 3.1** On \( X = \mathbb{R}^3 \) endowed with the norm

\[
\| (x_1, x_2, x_3) \| = \max \{ |x_1|, |x_2|, |x_3| \}.
\]

We consider the families of projection \( P_1, P_2, P_3 : \mathbb{R}_+ \rightarrow \mathcal{B}(X) \) defined by

\[
P_1(t)(x_1, x_2, x_3) = (x_1, 0, 0)
\]
\[
P_2(t)(x_1, x_2, x_3) = (0, x_2, 0)
\]
\[
P_3(t)(x_1, x_2, x_3) = (0, 0, x_3)
\]

for every \( t \geq 0 \) and \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). Given the increasing functions \( h, k : \mathbb{R}_+ \rightarrow [1, \infty) \) we consider the evolution operator \( U : \Delta \rightarrow \mathcal{B}(X) \) defined by

\[
U(t,s)(x_1, x_2, x_3) = \left( \frac{k(s)}{k(t)}, \left( \frac{h(s)}{h(t)} \right)^2 x_1, \frac{k(t)}{k(s)}, \left( \frac{h(t)}{h(s)} \right)^2 x_2, \frac{h(s)}{h(t)} x_3 \right)
\]

So

\[
U(t,s) = \frac{k(s)}{k(t)} \cdot \left( \frac{h(s)}{h(t)} \right)^2 P_1(s) + \frac{k(t)}{k(s)} \cdot \left( \frac{h(t)}{h(s)} \right)^2 P_2(s) + \frac{h(s)}{h(t)} P_3(s)
\]
and hence

\begin{align*}
    h(t)^2\|U(t, s)P_1(s)x\| &= \frac{k(s)}{k(t)} \cdot h(s)^2\|P_1(s)x\| \leq k(s) \cdot h(s)^2\|P_1(s)x\|, \\
    h(t)^2\|P_2(s)x\| &= \frac{k(s)}{k(t)} \cdot h(s)^2\|U(t, s)P_2(s)x\| \leq k(t)h(s)^2\|U(t, s)P_2(s)x\|, \\
    h(s)\|U(t, s)P_3(s)x\| &= \frac{h(s)^2}{h(t)}\|P_3(s)x\| \leq h(s)\|P_3(s)x\| \leq h(t)k(s)\|P_3(s)x\|, \\
    h(s)\|P_3(s)x\| &= h(t)\|U(t, s)P_3(s)x\| \leq h(t)k(t)\|U(t, s)P_3(s)x\|,
\end{align*}

for all \((t, s, x) \in \Delta \times X\).

So for all \((h, k)\) there exists an evolution operator \(U\) such that \(U\) is \((h, k)\)-trichotomic.

A first characterization of \((h, k)\)-trichotomy is given by

**Theorem 3.1** The trichotomic pair \((U, P)\) is \((h, k)\)-trichotomic if and only if there are \(N \geq 1, a > 0, b \geq 0, c > 0\)

\begin{align*}
    (t'_1) \quad h(t)^a\|U(t, t_0)P_1(t_0)x_0\| &\leq Nh(s)^a k(s)^b\|U(s, t_0)P_1(t_0)x_0\|; \\
    (t'_2) \quad h(t)^a\|U(t_0, t_0)P_2(t_0)x_0\| &\leq Nh(s)^a k(t)^b\|U(t, t_0)P_2(t_0)x_0\|; \\
    (t'_3) \quad h(s)^c\|U(t, t_0)P_3(t_0)x_0\| &\leq Nh(t)^c k(s)^b\|U(s, t_0)P_3(t_0)x_0\|; \\
    (t'_4) \quad h(s)^c\|U(t_0, t_0)P_3(t_0)x_0\| &\leq Nh(t)^c k(t)^b\|U(t, t_0)P_3(t_0)x_0\|,
\end{align*}

for all \(t \geq s \geq t_0 \geq 0\) and \(x_0 \in X\).

**Proof.** Necessity.
\( (t_1) \Rightarrow (t'_1) \) If we suppose that \( (t_1) \) holds then
\[
 h(t)^a\|U(t, t_0)P_1(t_0)x_0\| = h(t)^a\|U(t, s)P_1(s)U(s, t_0)P_1(t_0)x_0\| \leq \]
\[
 \leq Nh(s)^a k(s)^b \|P_1(s)U(s, t_0)P_1(t_0)x_0\| = \]
\[
 = Nh(s)^a k(s)^b \|U(s, t_0)P_1(t_0)x_0\|,
\]
for all \( (t, s, t_0, x_0) \in \mathbb{R}_+^3 \times X \) with \( t \geq s \geq t_0 \).

\( (t_2) \Rightarrow (t'_2) \) From \( (t_2) \) it follows
\[
 h(t)^a\|U(s, t_0)P_2(t_0)x_0\| = h(t)^a\|P_2(s)U(s, t_0)P_2(t_0)x_0\| \leq \]
\[
 \leq Nh(s)^a k(t)^b \|U(t, s)P_2(s)U(s, t_0)P_2(t_0)x_0\| = \]
\[
 = Nh(s)^a k(t)^b \|U(t, t_0)P_2(t_0)x_0\|,
\]
and hence \( (t'_2) \) is verified.

\( (t_3) \Rightarrow (t'_3) \) By \( (t_3) \) we have
\[
 h(s)^c\|U(t, t_0)P_3(t_0)x_0\| = h(s)^c\|U(t, s)P_3(s)U(s, t_0)P_3(t_0)x_0\| \leq \]
\[
 \leq Nh(t)^c k(s)^b \|P_3(s)U(s, t_0)P_3(t_0)x_0\| = Nh(t)^c k(s)^b \|U(s, t_0)P_3(t_0)x_0\|,
\]
for all \( t \geq s \geq t_0 \geq 0 \) and all \( x_0 \in X \).

\( (t_4) \Rightarrow (t'_4) \) The inequality \( (t_4) \) implies
\[
 h(s)^c\|U(s, t_0)P_3(t_0)x_0\| = h(s)^c\|P_3(s)U(s, t_0)x_0\| \leq \]
\[
 \leq Nh(t)^c k(t)^b \|U(t, s)P_3(s)U(s, t_0)x_0\| = Nh(t)^c k(t)^b \|U(t, t_0)P_3(t_0)x_0\|,
\]
for all \( t \geq s \geq t_0 \) and \( x_0 \in X \), which shows that \( (t'_4) \) holds.

**Sufficiency.** It is obvious, taking \( t_0 = s \) in \( (t_i) \) with \( i \in \{1, 2, 3, 4\} \). 

A characterization of \((h,k)\)-trichotomy in the particular case when the family \( \mathcal{P} \) is supplementary and compatible for \( U \) is given by

**Theorem 3.2** If \( \mathcal{P} \) is a supplementary family of projections which is compatible for \( U \) then the pair \((U, \mathcal{P})\) is \((h,k)\)-trichotomic if and only if there exist \( N \geq 1, a > 0, b \geq 0, c > 0 \) such that:

\( (t'_2) \) \( h(t)^a\|U(t, s)P_1(s)x\| \leq Nh(s)^a k(s)^b \|P_1(s)x\|; \)
\begin{align}
(t_2') \ h(t)^a\|V(t, s)P_2(t)x\| & \leq Nh(s)^a k(t)^b\|P_2(t)x\|; \\
(t_3') \ h(s)^c\|U(t, s)P_3(s)x\| & \leq Nh(t)^c k(s)^b\|P_3(s)x\|; \\
(t_4') \ h(s)^c\|V(t, s)P_3(t)x\| & \leq Nh(t)^c k(t)^b\|P_3(t)x\|,
\end{align}
for all \((t, s, x) \in \Delta \times X\).

**Proof.** It is sufficient to prove the equivalences \((t_2) \Leftrightarrow (t_2')\) and \((t_4) \Leftrightarrow (t_4')\).

**Necessity.**

\((t_2) \Rightarrow (t_2')\) By \((t_2)\) we have
\[
h(t)^a\|V(t, s)P_2(t)x\| = h(t)^a\|P_2(s)\|\|V(t, s)P_2(t)x\| \leq Nh(s)^a k(t)^b\|P_2(t)x\|,
\]
and hence \((t_2')\) holds.

\((t_4) \Rightarrow (t_4')\) From \((t_4)\) we have
\[
h(s)^c\|V(t, s)P_3(t)x\| = h(s)^c\|P_3(s)\|\|V(t, s)P_3(t)x\| \leq Nh(t)^a k(t)^b\|U(t, s)P_3(s)V(t, s)P_3(t)x\| = Nh(t)^a k(t)^b\|P_3(t)x\|,
\]
for all \((t, s, x) \in \Delta \times X\) and hence \((t_4')\) is proved.

**Sufficiency.**

\((t_2') \Rightarrow (t_2)\) If we suppose \((t_2')\) then
\[
h(t)^a\|P_2(s)x\| = h(t)^a\|V(t, s)U(t, s)P_2(s)x\| \leq Nh(s)^a k(t)^b\|U(t, s)P_2(s)x\|
\]
and hence \((t_2)\) is proved.

\((t_4') \Rightarrow (t_4)\) By \((t_4')\) it results
\[
h(s)^c\|P_3(s)x\| = h(s)^c\|V(t, s)U(t, s)P_3(s)x\| \leq Nh(t)^c k(t)^b\|U(t, s)P_3(s)x\|,
\]
for all \((t, s, x) \in \Delta \times X\) and hence \((t_4)\) holds.

\[\blacksquare\]
4 Strong (h,k)-trichotomy

Now we introduce a new concept of (h,k)-trichotomy.

**Definition 4.1** Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of three supplementary projections which is compatible with the evolution operator $U : \Delta \to B(X)$. We say that the pair $(U, \mathcal{P})$ is strongly (h,k)-trichotomic and denote $s \sim (h,k) - t$, if there are $N \geq 1, a > 0, b \geq 0, c > 0$ such that for all $(t, s, x) \in \Delta \times X$ the following properties hold:

\[
\begin{align*}
(st_1) & \quad h(t)^a \|U(t, s)P_1(s)x\| \leq Nh(s)^a k(s)^b \|x\|; \\
(st_2) & \quad h(t)^a \|V(t, s)P_2(t)x\| \leq Nh(s)^a k(t)^b \|x\|; \\
(st_3) & \quad h(s)^c \|U(t, s)P_3(s)x\| \leq Nh(t)^c k(s)^b \|x\|; \\
(st_4) & \quad h(s)^c \|V(t, s)P_3(t)x\| \leq Nh(t)^c k(t)^b \|x\|.
\end{align*}
\]

**Remark 4.1** If the function $k$ is constant or $b = 0$ then we obtain the concept of strong h-trichotomy.

**Remark 4.2** The pair $(U, \mathcal{P})$ is strongly (h,k)-trichotomic if and only if there are $N \geq 1, a > 0, b \geq 0, c > 0$ such that:

\[
\begin{align*}
(st'_1) & \quad h(t)^a \|U(t, s)P_1(s)\| \leq Nh(s)^a k(s)^b; \\
(st'_2) & \quad h(t)^a \|V(t, s)P_2(t)\| \leq Nh(s)^a k(t)^b; \\
(st'_3) & \quad h(s)^c \|U(t, s)P_3(s)\| \leq Nh(t)^c k(s)^b; \\
(st'_4) & \quad h(s)^c \|V(t, s)P_3(t)\| \leq Nh(t)^c k(t)^b \quad \text{for all } (t, s) \in \Delta.
\end{align*}
\]

**Definition 4.2** A family of projections $\mathcal{P} = \{P_1, P_2, P_3\}$ is called k-bounded if there exists $M \geq 1$ and $d \geq 0$ such that

\[
\|P_j(t)\| \leq Mk(t)^d,
\]

for all $t \geq 0$ and all $j \in \{1, 2, 3\}$. 
Remark 4.3 If the pair $(U, P)$ is strongly $(h, k)$-trichotomic then $P$ is $k$-bounded.

Remark 4.4 If the pair $(U, P)$ is strongly $(h, k)$-trichotomic then it is $(h, k)$-trichotomic.

**Proof.** Indeed, if we substitute $x$ with $P_1(s)x$ in $(st_1)$, $x$ with $P_2(t)x$ in $(st_2)$, $x$ with $P_3(s)x$ in $(st_3)$ respectively $x$ with $P_3(t)x$ in $(st_4)$ then we obtain that the conditions $(t_1), (t_2), (t_3)$ and $(t_4)$ are satisfied.  

Example 4.1 On $X = \mathbb{R}^3$ endowed with the norm

$$\|(x_1, x_2, x_3)\| = \max\{|x_1|, |x_2|, |x_3|\}.$$ 

Let $h, k : \mathbb{R}_+ \to [1, \infty)$ be two increasing functions.

We consider the families of projections give by $P = \{P_1, P_2, P_3\}$ with $P_1, P_2, P_3 : \mathbb{R}_+ \to B(X)$ defined by

$$P_1(t)(x_1, x_2, x_3) = (x_1 + e^{k_2(t)}x_2, 0, 0)$$

$$P_2(t)(x_1, x_2, x_3) = (-e^{k_2(t)}x_2, x_2, 0)$$

$$P_3(t)(x_1, x_2, x_3) = (0, 0, x_3)$$

for every $t \geq 0$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Consider the evolution operator $U : \Delta \to B(X)$ defined, for all $(t, s) \in \Delta$ by

$$U(t, s) = \frac{h(s)}{h(t)}P_1(s) + \frac{h(t)}{h(s)}P_2(t) + \frac{h(s)}{h(t)}P_3(s)$$

We have that,

$$h(t)\|U(t, s)P_1(s)x\| = h(s)\|P_1(s)x\| \leq k(s)h(s)\|P_1(s)x\|,$$

$$h(t)\|P_2(s)x\| \leq h(s)\|U(t, s)P_2(s)x\| \leq k(t)h(s)\|U(t, s)P_2(s)x\|,$$

$$h(s)\|U(t, s)P_3(s)x\| \leq h(s)\|P_3(s)x\| \leq h(t)k(s)\|P_3(s)x\|,$$
\[ h(s)\| P_3(s)x \| = h(t)\| U(t, s)P_3(s)x \| \leq h(t)k(t)\| U(t, s)P_3(s)x \| , \]

for all \((t, s, x) \in \Delta \times X\).

So \((U, P)\) is \((h, k)\)-trichotomic.

We observe that
\[ \| P_1(t)(0, 1, 0) \| = e^{k^2(t)} \]
and by on Remark 4.3 it results that \((U, P)\) cannot be strongly \((h, k)\)-trichotomic.

**Remark 4.5** The previous example shows that for every two increasing functions \(h, k : \mathbb{R}_+ \rightarrow [1, \infty)\) there exists a family of supplementary projections \(P = \{P_1, P_2, P_3\}\) and an evolution operator \(U\) such that

(i) \(P\) is compatible with \(U\);
(ii) \((U, P)\) is \((h, k)\) - trichotomic;
(iii) \((U, P)\) is not strongly \((h, k)\) - trichotomic.

5 Weak \((h,k)\)-trichotomy

We introduce a new concept of \((h,k)\)-trichotomy.

**Definition 5.1** Let \(P = \{P_1, P_2, P_3\}\) be a family of three supplementary projections which is compatible with the evolution operator \(U : \Delta \rightarrow B(X)\). We say that the pair \((U, P)\) is weakly \((h,k)\)-trichotomic and denote \(w \quad (h,k)\) - trichotomic if there are \(N \geq 1, a > 0, b \geq 0, c > 0\) such that for all \((t, s) \in \Delta\) the following properties hold:

\[ (wt_1) \quad h(t)^a\| U(t, s)P_1(s) \| \leq Nh(s)^a k(s)^b\| P_1(s) \| ; \]
\[ (wt_2) \quad h(t)^a\| V(t, s)P_2(t) \| \leq Nh(s)^a k(t)^b\| P_2(t) \| ; \]
\[ (wt_3) \quad h(s)^c\| U(t, s)P_3(s) \| \leq Nh(t)^c k(s)^b\| P_3(s) \| ; \]
\[ (wt_4) \quad h(s)^c\| V(t, s)P_3(t) \| \leq Nh(t)^c k(t)^b\| P_3(t) \|. \]

**Remark 5.1** If the pair \((U, P)\) is \((h,k)\)-trichotomic then it is weakly \((h,k)\)-trichotomic.
Remark 5.2 The connections between the trichotomy concepts considered in this paper are

\[ s - (h, t) - t \implies (h, k) - t \implies w - (h, k) - t \]

Remark 5.3 If \( \mathcal{P} = \{P_1, P_2, P_3\} \) is \( k \)-bounded then \( w - (h, k) - t \) implies \( s - (h, k) - t \).

A particular case when the trichotomy concepts considered in this paper are equivalent is given by

Remark 5.4 Let \( \mathcal{P} = \{P_1, P_2, P_3\} \) be a family of three supplementary projections which is \( k \)-bounded and compatible with the evolution operator \( U : \Delta \to \mathcal{B}(X) \). Then the following statements are equivalent:

(i) \( (U, \mathcal{P}) \) is strongly \((h,k)\)-trichotomic;

(ii) \( (U, \mathcal{P}) \) is \((h,k)\)-trichotomic;

(iii) \( (U, \mathcal{P}) \) is weakly \((h,k)\)-trichotomic.

Proof. By Remark 5.2, we have to prove that \( w - (h, k) - t \) implies \( s - (h, k) - t \). Let \( N \geq 1, a > 0, b \geq 0, c > 0 \) given by the \( w - (h, k) - t \) property and let \( M \geq 1, d \geq 0 \) be such that

\[ \|P_i(t)\| \leq Mk(t)^{d}, \]

for every \( t \geq 0 \) and \( i \in \{1, 2, 3\} \).

The conclusion follows from the bellow estimations:

- \[ h(t)^a \|U(t, s)P_1(s)\| \leq Nh(s)^a k(s)^b \|P_1(s)\| \leq MNh(s)^a k(s)^{b+d}, \]
- \[ h(t)^a \|V(t, s)P_2(t)\| \leq Nh(s)^a k(t)^b \|P_2(t)\| \leq MNh(s)^a k(t)^{b+d}, \]
- \[ h(s)^c \|U(t, s)P_3(s)\| \leq Nh(t)^c k(s)^b \|P_3(s)\| \leq MNh(t)^c k(s)^{b+d}, \]
- \[ h(s)^c \|V(t, s)P_3(t)\| \leq Nh(t)^c k(t)^b \|P_3(t)\| \leq MNh(t)^c k(t)^{b+d}, \]

for all \( t \geq s \geq 0 \).
References


